

A Pedagogical Approach to Quantum Computing using spin-1/2 particles

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Abstract: This paper discusses the important primitives of superposition and entanglement in QIP from physics of spin-1/2 particles. System of spin-1/2 particles present a logical and conceptual candidate to understand Quantum Computing. A pedagogical approach to abstract quantum information processing is considered in more concrete physical terms here.

1. INTRODUCTION

Quantum information processing (QIP) is set to change computation and communication in a way inconceivable by our classical perceptions. QIP derives its power, not from any revolutionary advances in the existing technologies, but from the logical system based on the quantum mechanical formalism. The quantum theoretic concepts on which QIP stands are superposition and entanglement.

Superposition refers to the possibility of a system existing as a linear combination of different states commensurate with the given conditions. It manifests in the form of observable interference effects between those states even in the particle description. Recall that in classical mechanics, interference is a characteristic of waves not expected of particles.

Entanglement refers to correlated states of two or more particles. There is nothing quantum mechanical about correlations per se for example, consider an experiment in which pairs of particles are created such that their total linear momentum is zero. Hence, if the momentum of one of the particles of a pair is found to be p in some direction then that of the other in that direction will certainly be found to be $-p$. This means that the value of the momentum of one particle determines completely that of the other i.e there is a correlation between their momenta. There is apparently nothing quantum mechanical about it. However, if remotely placed measuring devices are assumed to be uncorrelated then it turns out that certain characteristics of correlations between systems described quantum mechanically can not be reproduced by describing them by classical mechanics and probability theory. In fact, the simple example given above of pairs of particles created with zero total linear momentum was the one used by Einstein , Podolsky and Rosen (EPR) [1] to raise issues concerned with non classical features of correlations which are a matter of debate even today (see [2] and the reference therein for details).

In this article we intend to explain the concept of superposition and entanglement by considering the examples of one and two spin-1/2 systems(binary systems). We also intend to see the application of superposition and

entanglement in realm of biological systems. See also the articles in reference [3,5].

2. SUPERPOSITION IN SPIN-1/2 SYSTEMS

The dynamical state of a particle in classical mechanics is described by the values of its dynamical observables like position $q(t)$ and momentum $p(t)$ as a function of time. If the particle is moving in a potential of known form then, for given initial conditions, the values of the observables can be predicted with certainty, at least in principle. However, if the force acting on the particle is random then it may not be possible to predict with certainty the values of all of its observables. A familiar example of this situation is the motion of atoms constituting a gas. Every atom in a gas is subject to collisions with other atoms. The collisions are random giving rise to randomness in the motion of the atoms. The values of dynamical observables q, p of an atom in this case can not be evaluated deterministically. They are described statistically in terms of a probability distribution function $f(q,p)$ where $f(q,p)dqdp$ gives the probability that the position and momentum of an atom in the gas lie in the interval $q, q+dq$ and $p, p+dp$. The theory predicts average values of observables. The theoretical predictions are compared with experiments by carrying measurement of the observables on a large number of identical systems and by averaging the outcomes of such measurements.

In contrast with the classical theory, quantum theory does not assign definite values simultaneously to all the observables, like position and momentum, even in potentials of definite form. It makes only statistical predictions about the values of the observables. Hence, quantum theoretic predictions are compared with average result of identical experiments performed on identical systems.

In quantum theory, an isolated system is represented by the vector denoted, following Dirac, by the symbol $|\rangle$ called a *ket*. Thus a particular state of a system may be denoted by $|\psi\rangle$. The vector denoting a state is also referred to as a state vector. The set of all the vectors representing all the possible states of a system is said to constitute the *state space*. The Dirac notation may

be understood, for example, in the familiar language of vectors by thinking of the familiar vector notation \mathbf{a} or as a ket, say $|a\rangle$. In analogy with the familiar three-dimensional configuration space, the vector space of the states of a system is assigned the following properties:

- Like the *scalar product* $\mathbf{a} \cdot \mathbf{b}$ between two vectors \mathbf{a} and \mathbf{b} , we define scalar product between the vectors in the state space of the system. The scalar product between the vectors $|\psi\rangle$ and $|\phi\rangle$ is denoted by $\langle\psi|\phi\rangle$ where $\langle|\rangle$ is called a *bra*. The number $\langle\psi|\phi\rangle$ may be complex. The scalar product is defined so that $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle$
- The number $\sqrt{\langle\psi|\psi\rangle}$ is called the length or the norm of the vector $|\psi\rangle$.
- Two vectors $|\psi\rangle$ $|\phi\rangle$ are said to be orthogonal to each other if $\langle\phi|\psi\rangle = 0$
- Like any other vector in ordinary three dimensional space may be represented as a linear combination of three basis vectors, a vector in then may be represented in terms of a complete set of vectors called a *basis*. A basis is constituted by any set of linearly independent vectors in the state space. The number of linearly independent vectors in a given state is called its *dimension*.
- A basis may be constructed by identifying the possible values that an arbitrarily chosen observable of the given system can take. There may be more than one observables that can be measured simultaneously to any degree of accuracy. However, for the sake of definiteness, we assume that only one observable, say \hat{O} , can be measured at a time to any degree of accuracy. Let a_1, a_2, \dots, a_m be the complete set of values of \hat{O} with m as the dimension of the given state space. The number $a_i (i = 1, 2, \dots, m)$, being observables, are necessarily real. Let $|a_i\rangle$ be the state of the system when it is known to have the value $a_i (i = 1, 2, \dots, m)$ for the observable \hat{O} . It can be shown that the set of vectors $|a_1\rangle, |a_2\rangle, \dots, |a_m\rangle$

$$\langle a_i | a_j \rangle = \delta_{ij} \quad (1)$$

and that it is complete. Hence, any vector in the space can be expressed as a linear combination or superposition.

$$|\psi\rangle = \sum_i C_i |a_i\rangle \quad (2)$$

The set of vectors $\langle a_i | (i=1,2,\dots,m)$ thus constitutes a basis.

- On taking the scalar product of (2)with $\langle a_j |$ and using (1), the expansion coefficients C_i may be seen to be given by

$$C_j = \langle a_j | \psi \rangle \quad (3)$$

- If a state vector is represented as in (2) then its corresponding bra is represented as

$$\langle\psi| = \sum_{i=1}^m C_i^* \langle a_i | \quad (4)$$

- If (1) holds then, invoking also (4), it is straightforward to see that the scalar product between two vectors may be written as

$$\langle\phi|\psi\rangle = \sum_{i=1}^m D_i^* C_i \quad (5)$$

where

$$|\phi\rangle = \sum_{i=1}^m D_i |a_i\rangle \quad (6)$$

- If $|\psi\rangle$ represents a state and if c is a complex number then, according to the quantum theory, $c |\psi\rangle$ represents the same state. We may, therefore represent quantum states by the vectors whose norm is unity i.e by vectors $|\psi\rangle$ such that $\langle\psi|\psi\rangle = 1$. In what follows we will assume that the kets are normalized to unity. Hence if a state $|\psi\rangle$ is represented as in (2)then the normalization condition demands the relation

$$\sum_{i=1}^m |C_i|^2 = 1 \quad (7)$$

between the expansion coefficients. The complex number C_i in the expansion (2)is called the *probability amplitude* for the system to be in state $|a_i\rangle$. What is the physical significance of probability amplitude?

The physical significance of the probability amplitude is contained in the so called *measurement problem* of the quantum theory. According to it, if the possible outcomes of measurement of an observable on a system are a_1, a_2, \dots, a_m with the vectors $|a_1\rangle, |a_2\rangle, \dots, |a_m\rangle$ denoting the corresponding

states, and if the outcome of a particular act of measurement is, say a_i then after the measurement is over, the system goes over to or collapse to the state $|a_i\rangle$. Moreover, if we repeat the same experiment on several similarly prepared systems then the probability that the result of measurement is a_i is given by $|C_i|^2$. In other words, if we perform measurements of the observable \hat{O} on a number M of systems all of which are in the state given by (2) then the outcome of each measurement will be a number from the set of real numbers a_1, a_2, \dots, a_m . Let n_i be the number of systems which give a_i as the result of measurement. Then, in the limit $M \rightarrow \infty$,

$$|C_i|^2 = n_i/M \quad (8)$$

The average value of the observable \hat{O} then is

$$\langle \hat{O} \rangle = \sum_{i=1}^m a_i |C_i|^2 \quad (9)$$

The procedure outlined above, however, determines only the magnitude $|C_i|$ of the complex number

$$C_i = |C_i| \exp(i\theta) \quad (10)$$

but not its phase θ . Determination of phase requires phase sensitive measurement. However, we do not discuss that issue here.

It is thus clear that we can not determine the state of a system if we have only one copy of it. The determination of the state of a system requires a large collection of its identical copies. From this we conclude that we can not make copies of a quantum system if we do not know its state. For, if we have one copy of a quantum system in an unknown state and if we can make its copy then we can make any number of its copies and perform experiments on each of those copies to determine its state. That amounts to only one copy determining the state of the system! Though it is a trivial consequence of the measurement postulate of quantum theory, the impossibility of making a copy (or cloning) a quantum system in an unknown state is referred to as *no-cloning principle*.

We illustrate various notions introduced above by way of following two examples:

- Consider a particle whose spin measurement in any direction gives $\pm \hbar/2$ as the outcome. Consider a collection of such spins oriented in the direction \mathbf{e} . A measurement in the direction \mathbf{e} on any spin in that collection will give $\hbar/2$ as the answer. We label such a collection by the state $|1/2, \mathbf{e}\rangle$. Since $\pm \hbar/2$ are the only two possible outcomes of measurement of any spin component of the particle in question, its basis states are $|\pm 1/2, \mathbf{e}\rangle$ where \mathbf{e} is

an arbitrarily chosen direction. These basis states are orthogonal to each other i.e $\langle 1/2, \mathbf{e} | -1/2, \mathbf{e} \rangle = 0$. Consequently, the state of the spin in any direction can be represented in terms of the states $|\pm 1/2, \mathbf{e}\rangle$. The state space of the system in this case is, therefore, two dimensional.

A spin $1/2$ is a reality may be a two state system of interest in Nuclear Magnetic Resonance (NMR).

- Consider an atom with one valence electron. Let the energies of the levels that the electron can occupy be given by E_1, E_2, \dots . Any measurement of energy of the electron would yield one of those values of energy as the answer. If a measurement of the energy of the electron gives E_i as the answer then we say that the atom is in the state $|E_i\rangle$. The set of basis vectors in this case is $|E_1\rangle, |E_2\rangle, \dots$ which means that the dimension of the state space is countably infinite.

A particular case of interest is an atom constrained to make transitions between only two of its levels, say, the levels $|E_1\rangle$ and $|E_2\rangle$ having energies E_1 and E_2 ($E_2 > E_1$). That can be achieved if the atom, initially in a superposition of the states $|E_i\rangle$ and $|E_j\rangle$ interacts with a single frequency field ω such that $\omega \sim (E_2 - E_1)/\hbar$ and if for all the other energy levels E_i the detunings $|\omega - |E_i - E_1||/\hbar$ and $|\omega - |E_i - E_2||/\hbar$ are sufficiently large. The atom in this case is said to be a two level atom driven on resonance by the field.[2].

Whatever its realization, the state vector representing a two-level system may be written, following (2), as $|\psi\rangle = C_0|0\rangle + C_1|1\rangle$ where the states $|0\rangle$ and $|1\rangle$ are orthogonal to each other. The states $|0\rangle, |1\rangle$ may stand, for example, for the spin states $|\pm 1/2, \mathbf{e}\rangle$ or the two states $|E_1\rangle, |E_2\rangle$ of an atom involved in an interaction or the two states of any other two-level system. By virtue of (3), the expansion coefficients in (10) are given by

$$C_j = \langle j | \psi \rangle \quad (11)$$

where $j = 0, 1$

These coefficients are, of course, subject to the normalization condition (7).

3. SUPERPOSITION VERSUS MIXTURE

How is the probability arrived at by using the concept of probability amplitude different from the notion of probability in the classical theory of probability? Consider a two level atom in a superposition

state (10). As our discussion above indicates, that state represents a collection of identical atoms such that the valence electron in each of the atoms in that collection occupies the energy level E_0 with the probability $|C_0|^2$ and the energy level E_1 with the probability $|C_1|^2$. Does it not mean that electrons in a fraction $|C_0|^2$ of that collection occupy level E_0 and that the electrons in the remaining fraction $|C_1|^2$ occupy level E_1 ? The answer is: No. We can not separate the collection of atoms described by the state (10) into two parts in one of which the atoms are in one state and in the other state in the remaining part with the size of the two parts determined by the above mentioned probabilities. In other words, the expression (10) does not assign "either this" "or that" state to an electron. It states that the electron is "simultaneously" in the two states. Only when we perform an experiment to determine which energy state the electron is, in that we find the answer as E_0 or E_1 with the number of atoms in one or the other energy level determined by the probabilities $|C_0|^2$ and $|C_1|^2$. Of course, once an atom is experimentally found to be in a particular level, it remains in that level till disturbed. After the measurement to determine their energy is performed on all the atoms in the given collection, that collection separates in two groups. In one of the groups all the atoms are in the level E_0 and in the other all are in the level E_1 . If we mix these two groups then the probability of picking an atom from this collection such that it has energy E_0 is $|C_0|^2$ and that of picking an atom of energy E_1 is $|C_1|^2$. Though this is the same set of probabilities as for the superposed state (10), the collection of spins in question is represented, not by the state (10), but by the mixed state

$$\hat{\rho} = |C_0|^2|E_0\rangle\langle E_0| + |C_1|^2|E_1\rangle\langle E_1| \quad (12)$$

The operator $\hat{\rho}$ in the equation above is called the density matrix. However, here we do not discuss the concept of density matrix. Each atom in the collection described by (12) is in "either one" "or other" state.

If you feel uncomfortable with the counterintuitive picture of an electron being simultaneously in two levels (instead of being in either one or the other level) till an experiment throws it in one or the other level then you are not alone. You are in the company of none other than Einstein. He questioned this apparently peculiar situation by asking: "...does it mean moon is not there till I look at it"? and "God does not play dice". Howsoever strange it might appear, that is how the nature is in the eyes of the quantum theory!

One might wonder, after one finds the atom in one or the other level after performing an experiment, how does one know that it was "simultaneously" in the two levels and not in "either one" "or other" level till then? The answer is: we can not distinguish between "simultaneous" and "either" "/or" situation by an experiment that is designed to determine only the level in which the atom is! That is because such an experiment determines the magnitude $|C_0|$ and $|C_1|$ but not the phase of C_0 or that of C_1 . The said two situations can be distinguished by experiments which are sensitive to the phase.

4. CLASSICAL STATISTICS AND QUANTUM MECHANICS OF A SPIN-1/2 SYSTEM

In this section we examine the similarities and differences between the probabilistic predictions of quantum mechanics of a spin 1/2 and compare them with the predictions arrived at by its classical statistical description assuming that the spin is subject to a random force. In the following we take $\hbar = 1$.

Consider a collection of identically prepared spin-1/2 each oriented in the spherical polar direction (θ, ϕ) . Let $|+z\rangle$ denote the state of a collection of spins all in the direction $+z$ and let $| - z\rangle$ denote the state of a collection of spins all in the direction $-z$. Choosing these state as the basis, we may write [2]

$$|+; \theta, \phi\rangle = \cos(\theta/2)|+z\rangle + \sin(\theta/2)\exp(i\phi)|-z\rangle \quad (13)$$

On recalling (2) and the discussion following it, we see that a measurement of the z-component of spins in the state (13) would give 1/2 as the outcome with probability $\cos^2(\theta/2)$ and -1/2 as the outcome with probability $\sin^2(\theta/2)$. The average value of spin in the z-direction will therefore be $(\cos^2(\theta/2) - \sin^2(\theta/2)) = \cos(\theta)/2$

Now, let us visualize the spin-1/2 as a classical two-state object. In order to mimick the quantum results, we may assume that the spin is undergoing a random motion as a result of which its projection along any direction is not fixed but a random number which assumes the values $\pm 1/2$ with the same probability as the corresponding quantum spin. Thus, if we assume that the quantum spin in the direction (θ, ϕ) is represented by a classical spin whose projection along the direction (θ, ϕ) gives 1/2 as the outcome of measurement with probability $\cos^2(\theta/2)$ and -1/2 as the outcome of measurement

with probability $\sin^2(\theta/2)$ then the classical and the quantum pictures emerge in complete agreement.

There is, however, a catch in the argument above. For, recall from the Sec.2 that statistical aspects are brought in the classical picture if the system under consideration is under the influence of a random force due to its environment. However, the spin in our example here is isolated. It is not interacting with any environment. How do we explain its assumed random behavior? A possible way out is to assume that the spin interacts with a fictitious environment consisting of unknown entities called hidden variables. This picture is drawn in analogy with the motion of an atom in a gas undergoing random collisions with other atoms. In that case, the other atoms constitute hidden entities for an observer monitoring the motion of a particular atom.

Thus, if we assume the existence of a fictitious environment of hidden variables then the quantum mechanical prediction about the average value of a component of a spin-1/2 in any direction can be mimicked by treating the spin as a classical object. The average value of outcome of measurements of the component of a spin-1/2 in any direction thus does not provide any signature that can distinguish quantum and classical theories.

If the hidden variables theory succeeds in reproducing all those predictions of the quantum theory which are in agreement with observations then the problem of solving the mystery of quantum mechanics would reduce to solving the mystery of hidden variables. However, the system of two spin-1/2s, discussed in Sec.6 and 7 below brings out the shortcomings of the hidden variable theory.

The failure of the hidden variable theory, assuming the process of measurement to be local, is attributable to the property of entanglement discussed next.

5. ENTANGLEMENT

We have seen that an isolated quantum system is described by a set of probability amplitudes each for an admissible value of an observable or a set of observables which can be measured simultaneously to any degree of accuracy. Consider now a system consisting of two subsystems denoted by A and B. In principle, quantum theory permits measurement of any observable of A and any observable of B simultaneously to any degree of accuracy. Let a_1, a_2, \dots, a_m be the complete set of values of one of the observables of A with $|a_1\rangle, |a_2\rangle, \dots, |a_m\rangle$

denoting the corresponding states. Similarly, let b_1, b_2, \dots, b_n be the complete set of values of one of the observables of B with $|b_1\rangle, |b_2\rangle, \dots, |b_n\rangle$ denoting the corresponding states. Any state of the combined system is evidently described by the probability amplitudes C_{ij} for observing the set of mn amplitudes $(a_i, b_j)(i = 1, 2, \dots, m; j = 1, 2, \dots, n)$. The number $|C_{ij}|^2$ is the probability for A to have value a_i when the value of B is $b_j(i = 1, 2, \dots, m; j = 1, 2, \dots, n)$ as the outcome of the measurement of corresponding observables. The corresponding combined states are denoted by $|a_i, b_j\rangle \equiv |a_i\rangle|b_j\rangle$. A general combined state $|\psi\rangle$ is represented by

$$|\psi\rangle = \sum_{i,j} C_{ij} |a_i, b_j\rangle \quad (14)$$

Now, let the probability amplitudes C_{ij} be such that they reduce (14) to the factorized form

$$|\psi\rangle = |\psi_a\rangle|\psi_b\rangle \quad (15)$$

where $|\psi_a\rangle$ is a state of system A alone and $|\psi_b\rangle$ is the that of the system B alone. The form (15) implies that the systems A and B are uncoupled or uncorrelated.

However, if the combined state of the two systems A and B can not be factorized (as in (15)) into a product having a state only of A and a state only of B as its factors then it is called an entangled state. We discuss the significance of entangled states by means of the example of two spin-1/2s in the following sections.

6. TWO SPIN-1/2

Consider a system consisting of two spin 1/2s denoted by A and B. Following Sec.5 we can specify a state of the system in terms of values of the component of spin A along some direction \vec{e}_A and those of the components of spin B along the direction \vec{e}_B where \vec{e}_A and \vec{e}_B may be same or different directions. The possible states of such a system then are linear combinations of the states $|+\vec{e}_A, +\vec{e}_B\rangle, |+\vec{e}_A, -\vec{e}_B\rangle, |-\vec{e}_A, +\vec{e}_B\rangle, |-\vec{e}_A, -\vec{e}_B\rangle$ where $|\epsilon_a \vec{e}_A, \epsilon_b \vec{e}_B\rangle \equiv |\epsilon_a \vec{e}_A\rangle|\epsilon_b \vec{e}_B\rangle$ (where $\epsilon_a, \epsilon_b = \pm 1$) denotes the state of the combined system in which the components of spin A along the direction \vec{e}_A is $\epsilon_a/2$ and that of spin B along the direction \vec{e}_B is $\epsilon_B/2$. Any state of the combined system of two spin 1/2s may therefore be expressed as

$$\begin{aligned} |\psi\rangle = & \alpha_1 |+\vec{e}_A, +\vec{e}_B\rangle + \alpha_2 |+\vec{e}_A, -\vec{e}_B\rangle + \\ & \alpha_3 |-\vec{e}_A, +\vec{e}_B\rangle + \alpha_4 |-\vec{e}_A, -\vec{e}_B\rangle \end{aligned} \quad (16)$$

Clearly, $|\alpha_1|^2$ gives the probability that the result of simultaneous measurement of the component of spin A in the direction \vec{e}_A is $1/2$ and that of the component of spin B in the direction \vec{e}_B is also $1/2$. Similar interpretation applies to other terms in (16). The state (16) will be an entangled state if we can not write it in the form (15) with $|\psi_A\rangle$ representing a state of A alone. As an example of a non-entangled state, note that if $\alpha_i = 0$ for $i \neq 2$ then $|\psi\rangle$ in (16) is of the form (15) with $|\psi_A\rangle = |+\vec{e}_A\rangle$ and $|\psi_B\rangle = |+\vec{e}_B\rangle$. As another example, if $\alpha_i = 1/2$ for all i then $|\psi\rangle$ in (16) can be expressed in the factorized form (15) with $|\psi_A\rangle = (|+\vec{e}_A\rangle + |-\vec{e}_A\rangle)/\sqrt{2}$ and $|\psi_B\rangle = (|+\vec{e}_B\rangle + |-\vec{e}_B\rangle)/\sqrt{2}$. Consider now the state

$$|\psi\rangle = 1/\sqrt{2}[|\vec{e}, -\vec{e}\rangle - |-\vec{e}, \vec{e}\rangle] \quad (17)$$

It may be verified that this state can not be factorized as in (15). Hence it is an entangled state. Since the coefficient of $|\vec{e}, -\vec{e}\rangle$ and that of $|-\vec{e}, \vec{e}\rangle$ in (17) is zero, it follows that (17) represents a state for which the probability of finding both the spins aligned parallel or antiparallel to \vec{e} is zero.

The question that may be asked is: Can we measure the extent of entanglement? In other words, can we formulate a criterion according to which we can compare the extent of entanglement of two states and say that one state is more entangled than the other? It turns out that it is possible to construct a measure of entanglement of a state of a system to two spin-1/2s described in terms of a state vector as in (16). The state (17) is accordingly found to be maximally entangled (see [2] for details).

7. SIGNATURE OF NON-CLASSICALITY

In order to understand the meaning of non-classical features, consider the two spin-1/2s prepared in the maximally entangled state (17). Let those spins fly apart. After they are separated and do not interact any longer, measure the component of spin A in some direction \vec{a} and that of spin B in another direction \vec{b} (\vec{a} and \vec{b} are unit vectors). Repeat the experiment N times. These measurements give the numbers $N(+\vec{a}; +\vec{b})$, $N(+\vec{a}; -\vec{b})$, $N(-\vec{a}; +\vec{b})$, $N(-\vec{a}; -\vec{b})$ where $N(+\vec{a}; +\vec{b})$ is the number of times the result of the said joint measurement on A is $1/2$ when that on B is also $1/2$; $N(+\vec{a}; -\vec{b})$ is the number of times the result of the said joint measurement of A is $1/2$ when that on B is $-1/2$ and so

on. These measurements determine the probabilities $p(\epsilon_a \vec{a}; \epsilon_b \vec{b}) = N(\epsilon_a \vec{a}; \epsilon_b \vec{b})/N$ where $p(\epsilon_a \vec{a}; \epsilon_b \vec{b})$ is the probability that the outcome of joint measurement of the component of spin A in the direction \vec{a} is $\epsilon_a/2$ when that of the component of spin B in the direction \vec{b} is $\epsilon_b/2$ (with $\epsilon_a = \pm 1$, $\epsilon_b = \pm 1$). If the spins are in the state (17) then the quantum-theoretic expression for this probability is found to be given by [2]

$$p(\epsilon_a \vec{a}; \epsilon_b \vec{b}) = 1/4[1 - \epsilon_a \epsilon_b \vec{a} \cdot \vec{b}] \quad (18)$$

From this we infer that the probability of finding two spins in the same direction is zero, i.e if $\vec{a} \cdot \vec{b} = 1$ then $p(+\vec{a}; +\vec{b}) = p(-\vec{a}; -\vec{b}) = 0$. This implies that if a spin is found to be aligned in any direction \vec{a} then we know that the other spin is aligned along $-\vec{a}$. This is evidently consistent with the discussion following (17).

From the point of view of the discussion to follow, we consider the probabilities for the pairs of directions from a set of three directions \vec{a} , \vec{b} and \vec{c} . Use (18) to show that

$$p(+\vec{a}; +\vec{b}) + p(+\vec{b}; +\vec{c}) - p(+\vec{a}; +\vec{c}) = 1/2[\sin^2(\theta_{ab}/2) + \sin^2(\theta_{bc}/2) - \sin^2(\theta_{ac}/2)] \quad (19)$$

where θ_{ab} , θ_{bc} , θ_{ac} are the angles between the directions identified by the respective subscripts. The Eq.(19) is quantum theoretic prediction.

Let us now examine the probabilities on the left hand side of (19) by treating the two spins as classical two-state objects. In this picture, each of the spin component can assume the values $\pm 1/2$ with yet to be specified probabilities. An important identity in the classical description involving the probabilities appearing on the left hand side of (19) is found by expressing those probabilities in terms of joint probabilities in three directions as explained next. To that end, consider three directions \vec{a} , \vec{b} , \vec{c} and define a joint probability for the components of the two spins along those directions to have specific values. For example, we define the joint probability $p(\vec{a}, \vec{b}, \vec{c}; \vec{a}, \vec{b}, \vec{c})$ for the components of both the spins along the given three directions to be $1/2$ (the quantities to the left of the semicolon in p refer to spin A and those to its right refer to spin B). We can in general, define the joint probabilities $p(\pm \vec{a}, \pm \vec{b}, \pm \vec{c}; \pm \vec{a}, \pm \vec{b}, \pm \vec{c})$.

Note that the joint probabilities give the probability for components in different directions of the spins to have definite values simultaneously. We

can, of course, measure any component of one spin and the same or any other component of another simultaneously to any degree of accuracy. However, in quantum formalism, we can not assign definite values simultaneously to different components of same spin. Hence, there is no place for the joint probabilities for definite values of the components of a spin in different directions in quantum formalism. Hence, following the notation introduced above whereby the entries on two sides of the semicolon in p refer to different spin, $p(\pm\vec{a}; \pm\vec{b})$ is a legitimate quantum mechanical probability but any p in which there are more than one entries on any side of the semicolon is inadmissible in quantum theory. The consequences arrived at below by invoking the notion of joint probability are, therefore, strictly classical.

Next, recall that the probability $p(x_1, x_2, \dots, x_m)$ for m variables x_1, x_2, \dots, x_m may be obtained by summing (or integrating) over all the permissible values of the n variables $x_{m+1}, x_{m+2}, \dots, x_{m+n}$ in the probability $p(x_1, x_2, \dots, x_m, x_{m+1}, x_{m+2}, \dots, x_{m+n})$ of $m+n$ variables. Hence, the probabilities for two variables appearing on the left hand side of (19) may be expressed in terms of the joint probabilities for three directions introduced above. For example,

$$p(+\vec{a}; +\vec{b}) = \sum p(+\vec{a}, \epsilon_1\vec{b}, \epsilon_2\vec{c}; \epsilon_3\vec{a}, +\vec{b}, \epsilon_4\vec{c}) \quad (20)$$

where the summation is over the ϵ 's each taking the values ± 1 .

Now, we invoke (1) the key property of the entangled state (17) embodied in (18) namely that the probability of finding two spins aligned parallel to each other is zero, (2) the relations to the type (20) for the probabilities appearing in (19), and (3) the requirement that all the probabilities be non negative. These conditions lead to the inequality [2,4]

$$p(+\vec{a}; +\vec{b}) + p(+\vec{b}; +\vec{c}) - p(+\vec{a}; +\vec{c}) \geq 0 \quad (21)$$

called a *Bell's inequality*. Recall that the corresponding quantum theoretic result is the equality (19). That equality need not respect the inequality (21). A violation of (21) by (19) for some choice of directions would constitute a rebuttal of the classical description.

It is not difficult to see that if, for example, the three vectors are coplanar and $\theta_{ab} = \theta_{bc} = \pi/3, \theta_{ac} = 2\pi/3$, then the value of (19) is $-1/4$ which violates (21). Several other inequalities have been similarly derived for other situations and the experiments carried to confirm violation of Bell's inequalities.

Note from the discussion preceding (21) that of the three assumptions leading to the Bell's inequality, the first one concerns the preparation of the system in a particular state. The fact that such a state is a practical reality, the violation of (21) must be due to the failure of one or both of other two assumptions. The second assumption is based on the possibility of defining joint probabilities for different components to have definite values. We may, therefore, attribute the violation of (21) as confirming that the joint probabilities for different components of a spin to have definite values is indeed inadmissible. However, if we insist on retaining the concept of joint probabilities of all kinds then the only reason for the failure of (21) must be the failure of the assumption (3) which states that the probabilities must be positive. Its failure implies that the probabilities are not necessarily positive. The concept of negative probabilities, however, has no place in the classical theory. Hence, violation of a Bell's inequality in any case is a signature of non-classicality.

An entangled state thus exhibits features having no classical counterpart. Note that if the state is not entangled i.e if it can be factorized as in (15) then each of the spin behaves independently of the other. We have seen in Sec.4 that a single spin-1/2 does not exhibit any non-classical property as regards the averages of spin components. Hence two independent spins can not exhibit any non classical property implying thereby that a factorizable spin state may be mimicked classically. Recall also that though the quantum averages of a single spin in a superposition state can be reproduced by its classical statistical description, that description can not reproduce the properties resulting from interference between the two states.

The pillars of QIP, namely superposition and entanglement thus can not be built by any classical prescription!

8. APPLICATION TO QUANTUM LIFE UNITS

If we want to go into a deep understanding of the phenomenon of superposition and entanglement, the question that comes forth is: Whether this phenomenon of super position and entanglement is restricted to quantum domain or it plays an important role in the dynamics of living units. Inanimate objects in macroscopic world are totally deterministic, the probability arising in the macroscopic world is only due to the lack of knowledge of the system. If we consider a simple example of coin tossing the chance factor that comes into play because we

are totally ignorant of the parameters like air resistance, bias ness of the coin, etc. On the contrary quantum mechanics, representing the microscopic system, is totally probabilistic, and this probability is not because of the lack of the knowledge of the system but fundamentally due to the uncertainty in the system given by Heisenberg's Uncertainty principle.

If we analyze into the dynamics of living units, what is then the underlying physics of living units, 'Is it classical mechanics or quantum mechanics?' Even if we try to find out a correspondence between the dynamics of the living system with that of the non living systems, still there remains some extensive biological processes which are difficult to correlate with the quantum or classical world. Though the processes like replication, culling of living species find different interpretation in quantum world, but still there are many processes which are not explained in terms of quantum theory. Similarly it is important to ask if physical phenomenon like superposition and entanglement have certain meaning in biological world or not.

Even if we assume that the phenomenon of superposition is permissible in the biological domain, then how are we going to explain the phenomenon of cloning which is allowed for living units? This can be probably understood as follows: In the quantum world the initial state of the system is $|\psi(0)\rangle = |\xi\rangle$ then it undergoes evolution in a linear super position of a basis states such as $\sum_{\xi} \psi_{\xi}(l)|\xi\rangle$ and after a certain time period $t = T$ it comes back to the original state. Then the quantum states representing artificial living systems can be copied at certain time periods like $t = 0, T, 2T, \dots$ and so on. If the living organisms are microscopically small and if quantum states are represented as artificial living systems, superposition do prevail at this level of the living organisms [5].

Regarding another important non classical feature of 'Entanglement', this can be related with the property of interdependency between different species which is shown in [5], process of entangling the several copies of the living organisms with one of the mutated version.

9. CONCLUSION

Thus pillars of QIP namely superposition and entanglement play an important role in our understanding of physics at the quantum scale. Its application in areas like information processing and biological systems carries a lot of promise for technological progress and conceptual understanding. The pedagogical understanding of QIP primitives by physical spin-1/2 systems gives a new outlook to understand how quantum information derives its meaning from physics of the quantum. Also it motivates us to look for more physical systems as resources in information processing.

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